Graphs - Grafikler

HTE - Temmuz 2018

1. Basics

Graph is a way to illustrate the behaviour, dependence of a function against a certain variable or variables. In this representation, the convention for a single variable is as follows

$$\begin{array}{l} \text{Function symbol : } s \\ \downarrow \\ s(t) \;\leftarrow\; \text{Function itself} \\ \;\;\uparrow \\ \text{Variable symbol : } t \end{array} \tag{1.1}$$

There may be cases where the function **strongly** depends on its variable or it depends **weakly** on its variable. These are respectively depicted in Fig. 1.1.

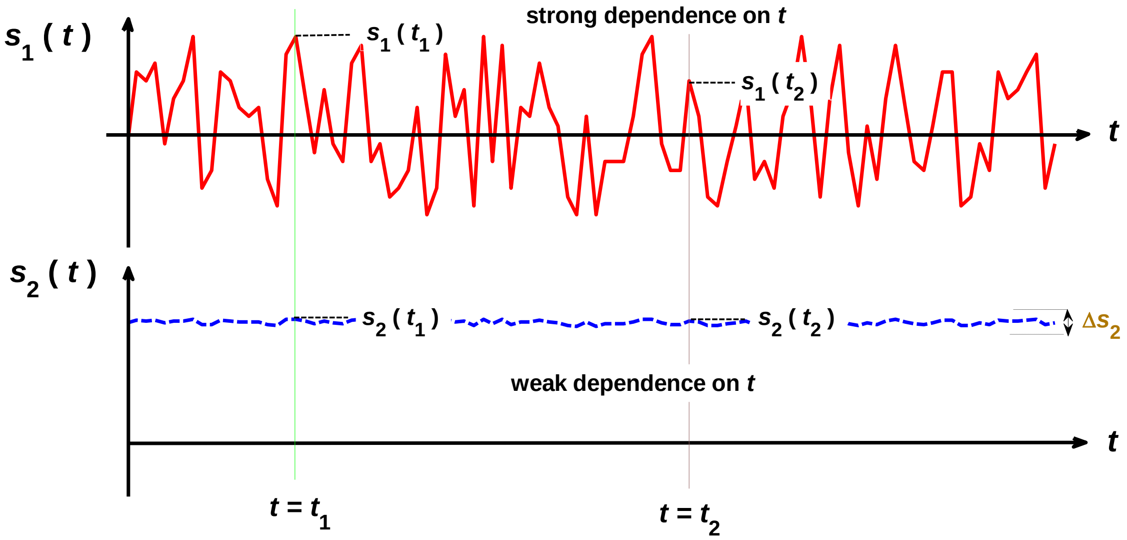


Fig. 1.1 Illustration of two types of functional dependence on its variable.

It is easy to conclude from Fig. 1.1 that

$$s_1(t_1) \neq s_1(t_2) \quad , \quad s_2(t_1) \simeq s_2(t_2) \;\text{ or }\; \Delta s_2 \to 0 \tag{1.2}$$

and (1.2) will be valid almost whichever t_1 and t_2 are taken. This means $s_1(t)$ varies quite a lot with t, while $s_2(t)$ is nearly invariant or hardly varies with t. Thus we classify that for $s_1(t)$, there is strong dependence on t, while for $s_2(t)$, there is weak dependence on t.

Bearing in mind the definition in (1.1), we can say that in Fig. 1.1, the horizontal axis is chosen to represent the variable, t while the vertical axis corresponds to the function itself, i.e. $s_1(t)$ or $s_2(t)$.

In some cases, the construction of a function is made from experimental data, i.e. measurements. Such a case is illustrated in Fig. 1.2.

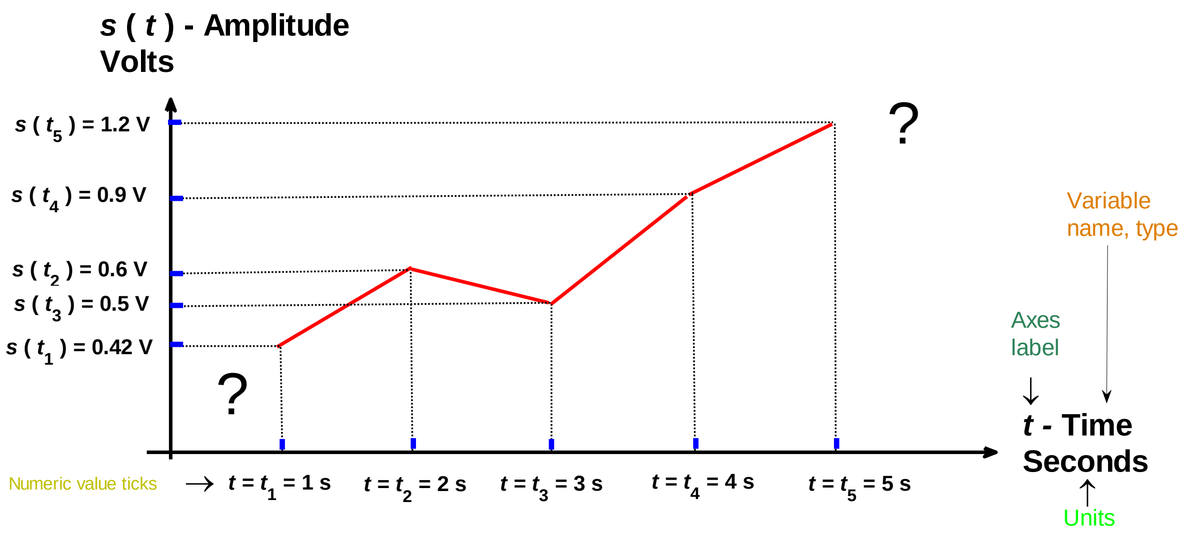


Fig. 1.2 Constructing the graph of a function from experimental data.

The appearance in Fig. 1.2 implies that we have performed our experiment at the times of $t = t_1 = 1$ s, $t = t_2 = 2$ s, $t = t_3 = 3$ s, $t = t_4 = 4$ s, $t = t_5 = 5$ s and obtained the functional numeric values of $s(t_1) = 0.42$ V, $s(t_2) = 0.6$ V, $s(t_3) = 0.5$ V, $s(t_4) = 0.9$ V, $s(t_5) = 1.2$ V. Here since the graph in Fig. 1.2 is plotted from experimental data, we have indicated both on the axes and for the measurements, the symbolic naming of the parameters, numeric values and units. Such a notation is important from the clarity of the graph. It is possible to determine the behaviour of $s(t)$ prior to $t = 1$ s (seconds) and after $t = 5$ s by taking measurements in those time ranges. At the moment, this is not done, so in Fig. 1.2, those regions are marked with ?. Furthermore, it is possible to take intermediate measurements so the time intervals between, t_1 and t_2, t_2 and t_3, t_3 and t_4, t_4 and t_5 are better represented (sampled). This way, we could obtain a smoother curve instead of the one in Fig. 1.2 which is in the form of straight line.

The functions plotted in Figs. 1.1 and 1.2, have only one variable, thus a two dimensional graph is sufficient for illustration. Now we turn to graphs of functions with two variables. Such an example is given in Fig. 1.3. There are two different plots in this figure, hence two functions, namely

First variable

$\downarrow$

$I_{sN}\left(s_x, s_y\right) \leftarrow$ First (upper figure) function

$\uparrow$

Second variable

First variable

$\downarrow$

$$I_{rN}\left(p_x, p_y\right) \leftarrow \text{ Second (lower figure) function} \tag{1.3}$$

$\uparrow$

Second variable

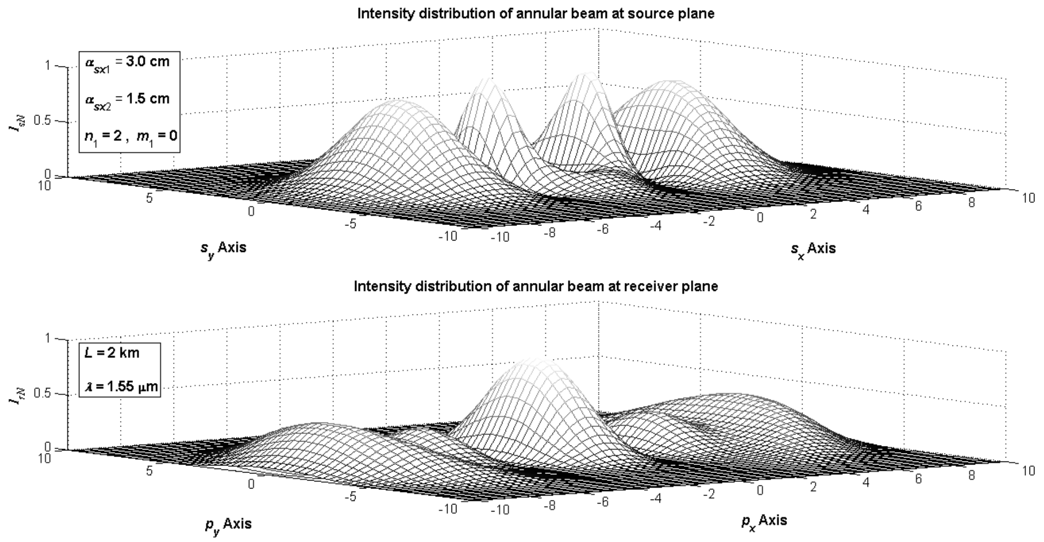


Fig. 1.3 An example of graphs of functions with two variables.

Since, there are two variables in the functions of Fig. 1.3, we have had to place two variables as horizontal axes and placed the function itself on the vertical axis. This way Fig. 1.3 is three dimensional as opposed to Figs. 1.1 and 1.2, which are two dimensional.

Another way of illustrating the graphs of Fig. 1.3 is to use what is called contour plot, where the values of the function are placed on the contour lines as shown in Fig. 1.4.

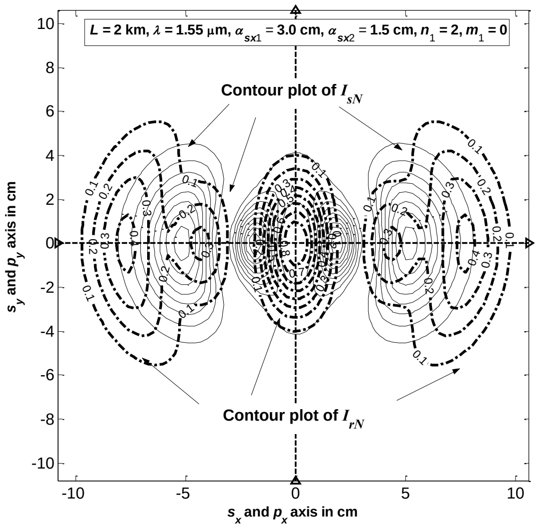


Fig. 1.4 Graphs of Fig. 1.3 in the form of contour plots.

It is clear from Figs. 1.3 and 1.4 that the functions $I_{sN}\left(s_x, s_y\right)$ and $I_{rN}\left(p_x, p_y\right)$ actually have dependencies more than two. As written in the legend boxes, these variable are set to constant values, for instance in the upper figure of Fig. 1.3, there are the settings of

$$\boxed{\begin{aligned} &\alpha_{sx1} = 3.0 \text{ cm} \\ &\alpha_{sx2} = 1.5 \text{ cm} \\ &n_1 = 2,\ m_1 = 0 \end{aligned}} \tag{1.4}$$

which means that if the numeric settings in (1.4) were to be modified, then the graphs of Figs. 1.3 and 1.4 would also change. It is also possible in some cases, to retain the two dimensionality of the graph and introduce the second variable as different curves at selected settings. Such an option is given in Fig. 1.5.

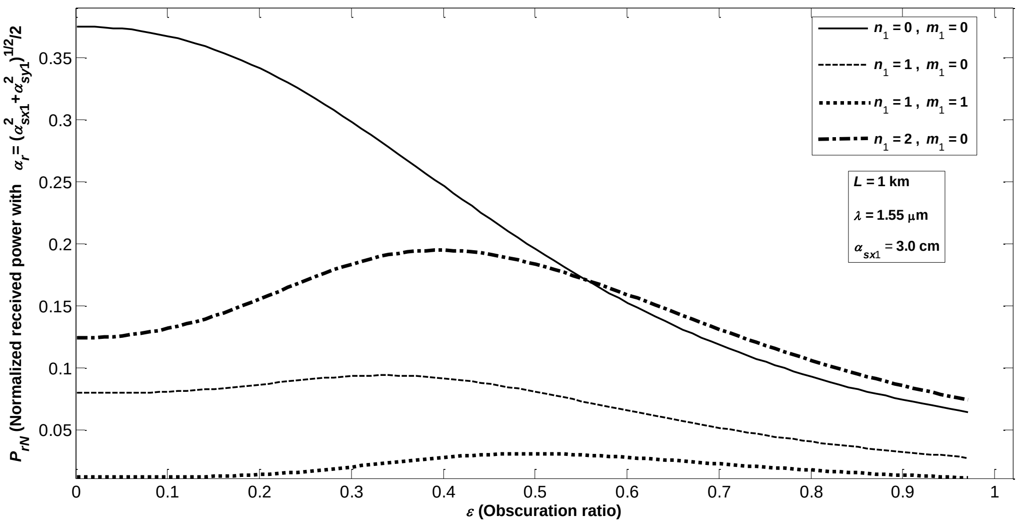
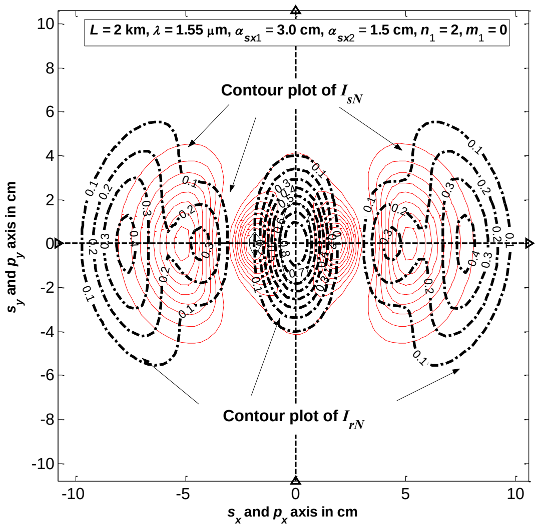


Fig. 1.5 A two dimensional graph containing the second variable as different curves.

From the examination of Figs. 1.1 to 1.5, we can summarize the type of information that should be present in a graph as follows.

a. In general, a graph should contain all the relevant information required in utmost clarity, avoiding all possible ambiguities.
b. The axes should be labelled to indicate which variable or function they refer to. If appropriate, the variable or function names, units should be provided on the axes. The axes should contain numeric value ticks. In this context, in the upper and lower figures of Fig. 1.3, the units of the two horizontal axis are missing, they are stated as cm (centimetre) in the contour plot of Fig. 1.4.
c. To emphasize the differentiation, line style differences, marker differences, colour differences should be incorporated. In this sense, the curves of Fig. 1.5 are clearly distinguishable, but it would be better, if the contour curves in Fig. 1.4 were to be distinguished in colour as shown below in Fig. 1.6.



2. Specific Functions and Time Axis Dilemma

In electronics and communication disciplines, some specific types of functions corresponding to representation of certain types of signals (waveforms) along time axis are of significance. Here we study two of them, called rectangular function and delta (impulse) function. They are shown in Fig. 2.1.

From Fig. 2.1, it is possible to write the mathematical expressions of rectangular and delta (impulse) functions as

Rectangular function

$$s_1(t)=\begin{cases} A & \text{for} \quad 0\le t\le T \\ 0 & \text{elsewhere} \end{cases} \quad \text{or} \quad s_1(t)=\begin{cases} A & \text{for} \quad 0\le t\le T \\ 0 & \text{for} \quad t<0 \text{ and } t>T \end{cases}$$

Delta function

$$s_2(t)=\begin{cases} \delta(t) & \text{for} \quad t=0 \\ 0 & \text{elsewhere} \end{cases} \quad \text{or} \quad s_2(t)=\begin{cases} \delta(t) & \text{for} \quad t=0 \\ 0 & \text{for} \quad t\ne 0 \end{cases} \tag{2.1}$$

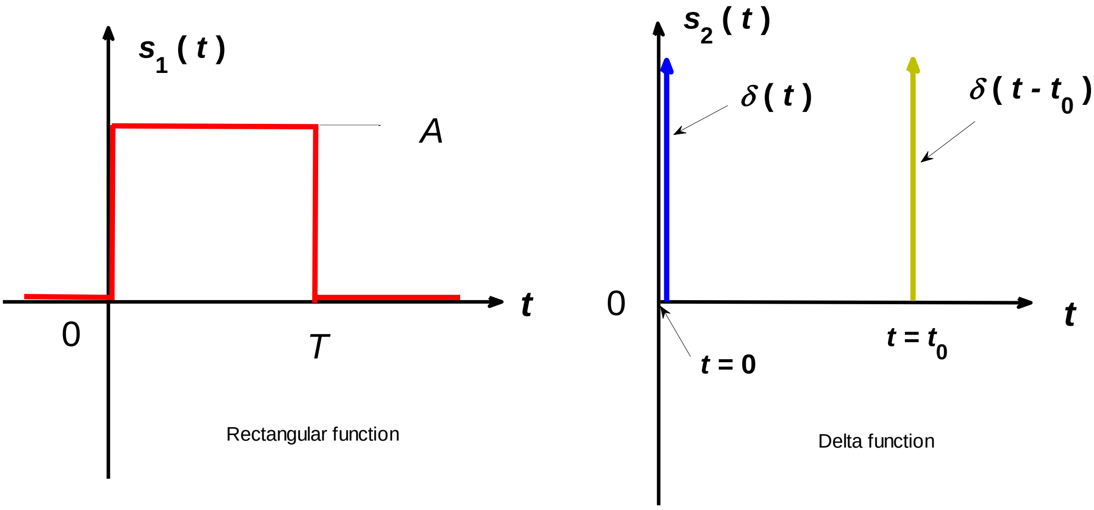


Fig. 2.1 Graphs of rectangular and delta functions.

From (2.1) and Fig. 2.1, we understand that $s_1(t)$ equals A for $0\le t\le T$ and zero outside this interval. It is important to realize that this $s_1(t)$ is completely different from

$$s(t)=A \qquad -\infty\le t\le\infty \tag{2.2}$$

In this sense, $s_1(t)$ is independent of its variable t, i.e. it is constant, when considered in the time interval of $0\le t\le T$, similarly $s_1(t)$ is also independent of its variable t outside the interval of $0\le t\le T$. It is only at time instances of $t=0$ and $t=T$, there are sudden changes in $s_1(t)$ from 0 to A and the other way around, respectively. But $s(t)$ is totally different, of course. It is independent of its variable t all throughout the time range. For comparison, we show the graph of $s(t)$ in Fig. 2.2.

Now coming onto $s_2(t)$ given on the second line of (2.1) and depicted on the left hand side of Fig. 2.1, we see that the two delta functions, $\delta(t)$ and $\delta(t-t_0)$ given here only exist (have nonzero value) at $t=0$ and $t=t_0$. Outside these instances, the delta functions are zero.

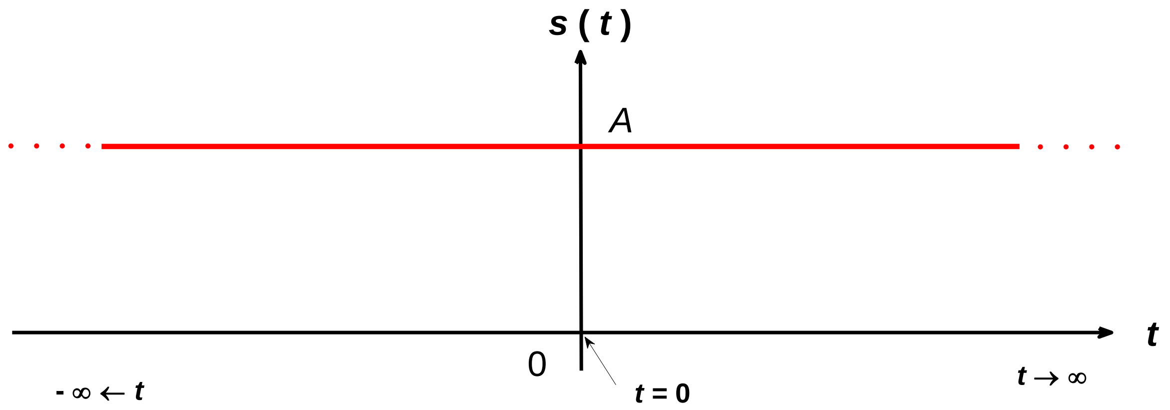


Fig. 2.2 Graph of $s(t)$ defined in (2.2).

It is possible to visualize that the delta function can be obtained from a rectangular function. For this we take rectangular function (pulse) of DC value unity and continue to reduce its amplitude as shown in Fig. 2.3.

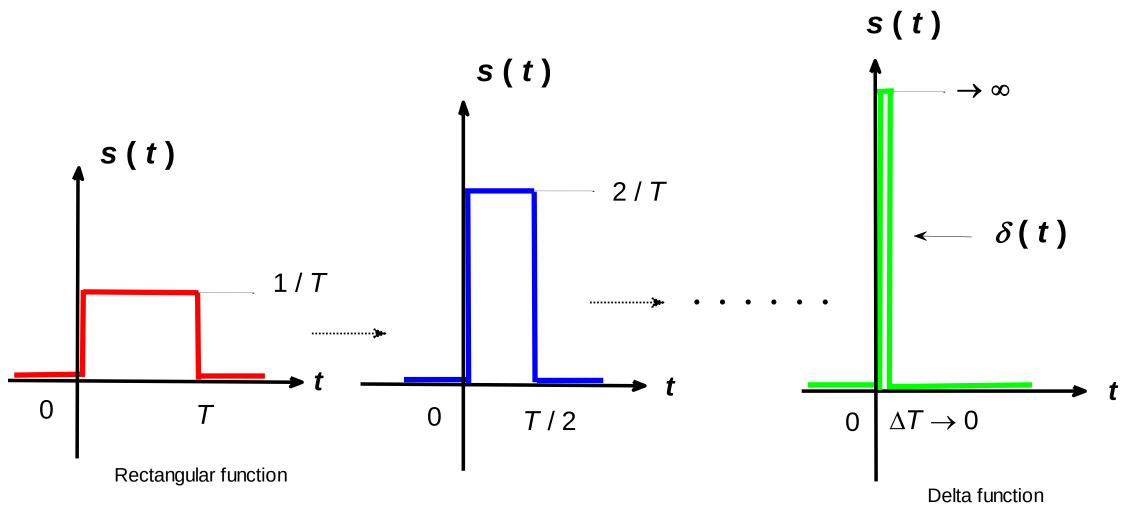


Fig. 2.3 The way of obtaining delta function from rectangular function.

As seen from Fig. 2.3, we arrive at the delta function by compressing the duration of rectangular pulse, thus to maintain the same DC value, the amplitude is increased, hence in the end we get

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{2.3}$$

Finally we remark that $\delta(t-t_0)$ is a time shifted version of $\delta(t)$, such that $\delta(t-t_0)$ has nonzero existence at $t = t_0$.

It is also possible to imagine the time periodic version of the rectangular function which is displayed in Fig. 2. 4.

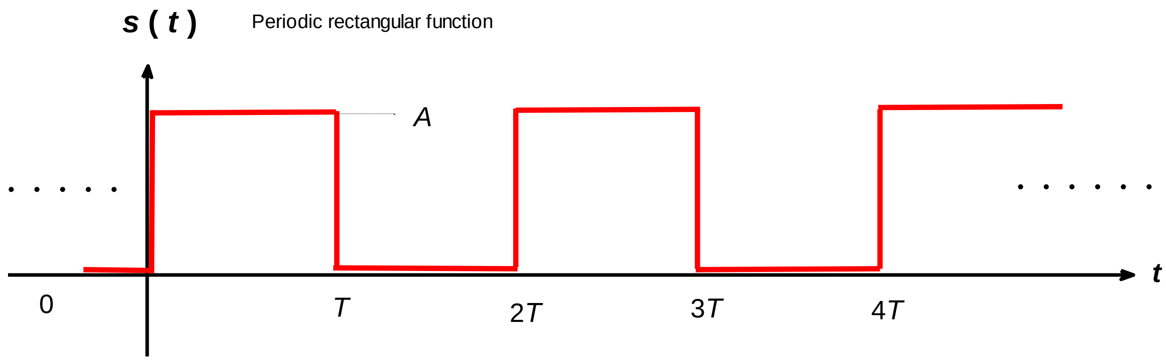


Fig. 2.4 Periodic rectangular function.

The mathematical expression for the periodic rectangular function of Fig. 2.4 can be written as

$$s(t) = A\sum_{n=-\infty}^{\infty} a_n U(t - nT) \tag{2.4}$$

where

$$a_n = \{0,\ 1\},\ U(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{2.5}$$

The writing in (2.5) means that a_n can only take two values, these are zero an done. $U \;\; t - nT$ is just the time shifted version of $U(t)$. $U(t)$ on the other hand is similar to $s_1(t)$ in Fig. 2.1.

Exercise 2.1 : Draw the graph of the following periodic function – Aşağıda verilen periyodik fonksiyonun grafiğini çizin.

$$s(t) = A\sum_{n=-\infty}^{\infty} a_n U(t - nT)$$

$$a_n = \{-1,\ 1\},\ U(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases} \tag{2.6}$$

Now, we want to talk about the time dilemma. In Fig. 2.5, we show the input into a system, where the input is the sliced version of periodic rectangular function.

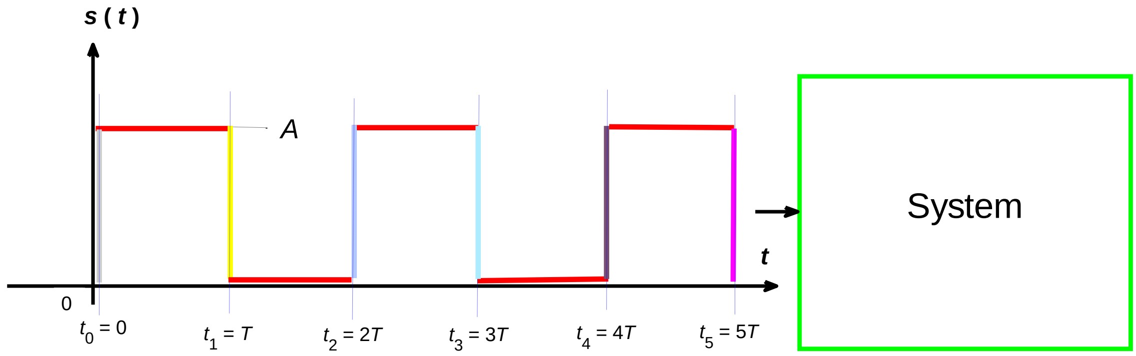


Fig. 2.5 The illustration of sliced rectangular input to a system with the accepted time convention.

Looking at Fig. 2.5, we witness something wrong, something strange. That is with the selection of the direction for the time axis of $s(t)$, we have the following order amoung the time quantities of Fig. 2.5

$$t_0 < t_1 < t_2 < t_3 < t_4 < t_5 \tag{2.7}$$

(2.7) means that along time axis, that t_0 is the earliest, t_1 is the next and so on, while t_5 is the latest. In other words, for instance t_1 is future with respect to t_0. We know that in order to be in line with physical reality, the $s(t)$ of Fig. 2.5 should enter the system with earliest part of the signal first, then the latest part of the signal last. The way to prevent and correct the wrong entry in Fig. 2.5 is displayed in Fig. 2.6.

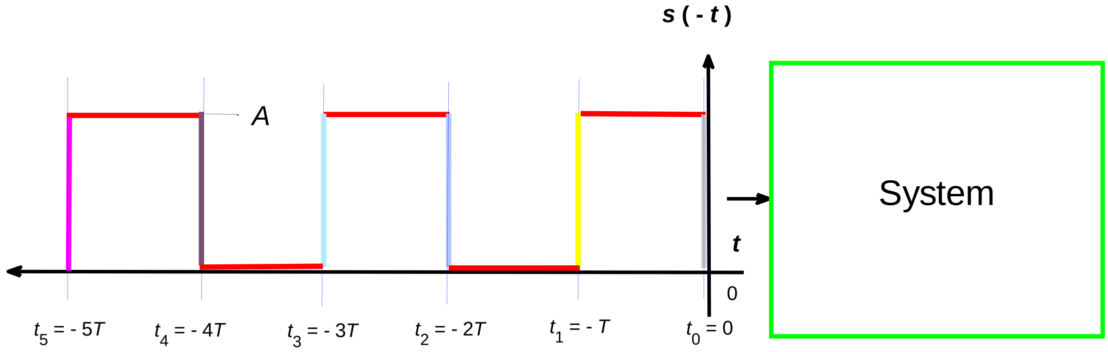


Fig. 2.6 The illustration of signal entry to a system in the correct way.

As seen from Fig. 2.6, the correct entry is established by time reversing the function $s(t)$, i.e. converting $s(t)$ into $s(-t)$. This way, the part of the signal that is the earliest in time, i.e. the part at t_0 enters the system first and the part of the signal that is the latest in time, i.e. the part at t_5 enters the system last.

What is described above is known as the time axis dilemma or rather the dilemma in the direction convention of the time axis. In our discipline, this is solved in the convolution integral by time reversing the input signal as done above, or by time reversing the system (physically difficult to grasp, but mathematically equivalent to the first case).

3. Differentiation of a Function

By differentiating a function, we arrive at its derivative, which is shown as

$$\frac{d}{dt}s(t) = \overset{\overset{\text{Prime}}{\downarrow}}{s'(t)} \tag{3.1}$$

The left hand side of (3.1) implies without any ambiguities that, the differentiation is taken with respect to the variable t. The right hand side of (3.1) is an equivalent notation for differentiation. But sometimes, it may be inconvenient to use prime, for instance if there were two variables of the function, it would not be clear with respect to which variable the differentiation is performed.

It should be stated that by taking the derivative of a function, we actually find the slope of the function, i.e. the tangent lines to the function at all different points of its variable. Hence differentiation reveals how rapidly or slowly a function varies with respect to its variable at those points. For a function of single variable, this is illustrated in Fig. 3.1 in a discrete manner for two points.

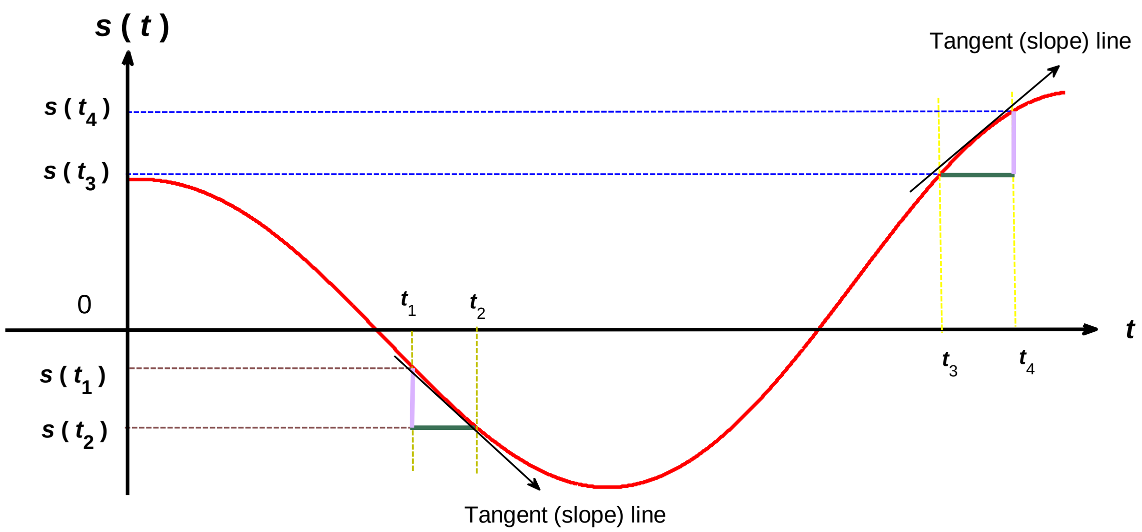


Fig. 3.1 Graphical illustration of the derivatives at two points.

The way of calculating the derivatives numerically from Fig. 3.1 is shown below.

$$\text{Approximated derivative (slope) of } s(t) \text{ in the time range of } t_1 \text{ and } t_2$$
$$\frac{d}{dt}s(t)\bigg|_{t_1 \Leftrightarrow t_2} = \frac{s(t_1)-s(t_2)}{t_1-t_2} \quad \rightarrow \frac{\Delta s}{\Delta t}$$
$$\text{Approximated derivative (slope) of } s(t) \text{ in the time range of } t_3 \text{ and } t_4$$
$$\frac{d}{dt}s(t)\bigg|_{t_3 \Leftrightarrow t_4} = \frac{s(t_3)-s(t_4)}{t_3-t_4} \quad \rightarrow \frac{\Delta s}{\Delta t} \tag{3.2}$$

As seen from Fig. 3.1, the derivative (slope) around t_1 and t_2 is negative (since $s(t_1)-s(t_2)$ is positive, but $t_1 - t_2$ is negative), while the derivative (slope) around t_3 and t_4 is positive. As we let t_1 approach t_2 or t_3 approach t_4, then we get the following

$$\frac{\Delta s}{\Delta t}\bigg|_{\Delta t \to 0} \rightarrow \frac{ds}{dt} = \frac{d}{dt}s(t) \tag{3.3}$$

The writing in (3.3) means that as Δt goes to zero, we obtain the analytic (mathematical) equivalent of the derivative which is independent of the time variable t. Thus, the expression $\frac{d}{dt}s(t)$ can be evaluated at any t, to find the derivative of $s(t)$ at that point.

We have formulated the derivative of the function in Fig. 3.1, intuitively assuming that the explicit mathematical formulation of $s(t)$ is unknown or too complex so that we have resort to numerical evaluation of the derivative. Now we turn to simple cases, where $s(t)$ is some simple analytic expression. One such example is given in (3.4)

$$s(t) = A + t^2 \quad , \quad \frac{d}{dt}s(t) = 2t \tag{3.4}$$

The plots of $s(t)$ and $\frac{d}{dt}s(t)$ are given in Fig. 3.2.

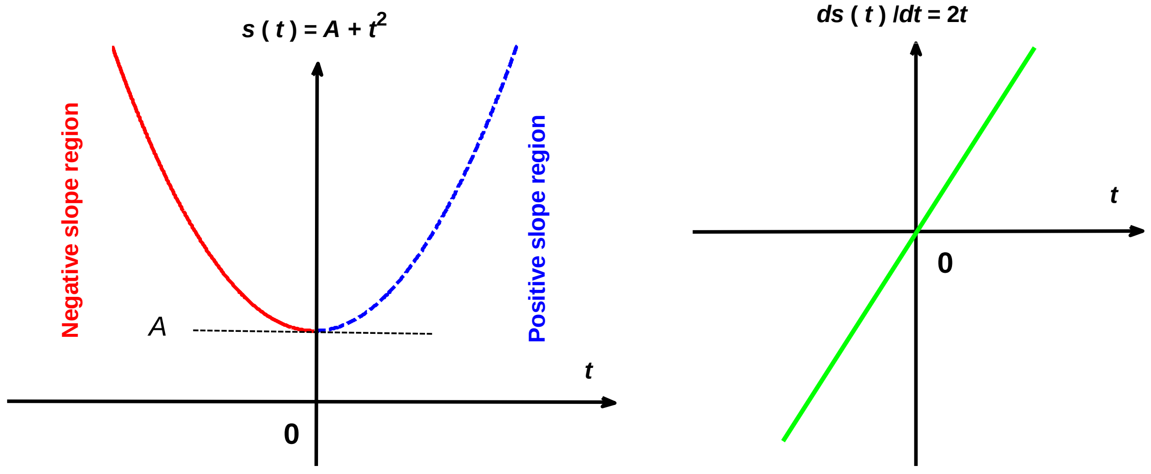


Fig. 3.2 The plots of $s(t)$ and $\dfrac{d}{dt}s(t)$ given in (3.4).

We can further differentiate the derivative of $s(t)$ to get the second derivative as follows

$$\frac{d}{dt}\left[\frac{d}{dt}s(t)\right]=\frac{d^2}{dt^2}s(t)=s''(t)=2 \qquad (3.5)$$

The plots of first and second derivatives of $s(t)$ are depicted in Fig. 3.3.

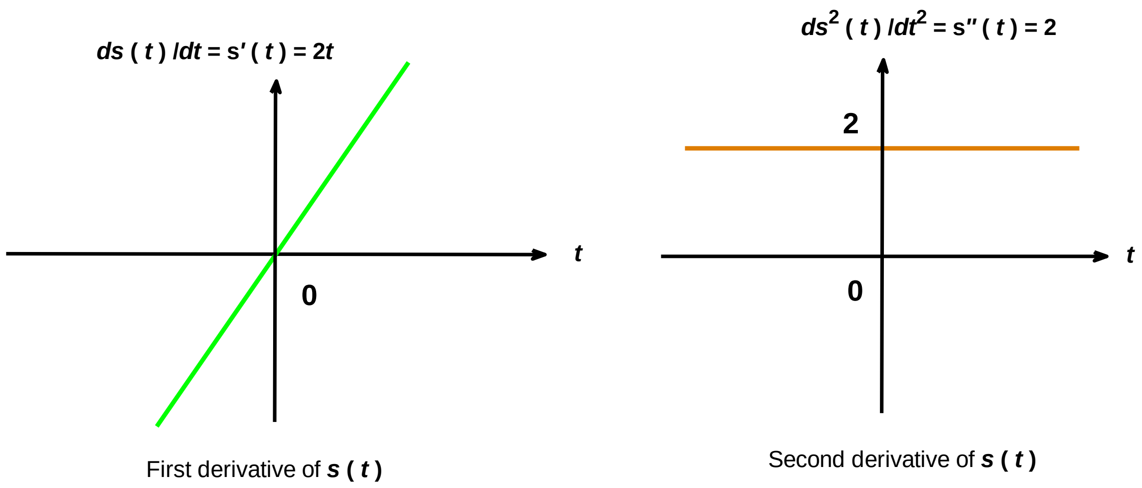


Fig. 3.3 The plots of first and second derivatives of $s\ \ t$.

Note that (3.5) is performed in Matlab by the following simple commands (in file named, simple_diff.m)

```
clear;clc;close all
syms A t st  %%%%% declaring A t and st symbolically
```

```
st = A + t^2;  %%%%% Writing for s(t)
stp = diff(st,t,1) %%%% First derivative s'(t)
stp2 = diff(st,t,2) %%%% Second derivative s''(t)
t = -2:0.1:2; A = 1;%%%% Giving a numeric range for t
%%%%% Plot section
st = eval(st);
stp = eval(stp); %%%%% Finding numeric values of first derivative
stp2 = repmat(stp2,length(t)); %%%%% Finding numeric values of second derivative
plot(t,st,'-k','LineWidth',2);hold on
plot(t,stp,'--r','LineWidth',2);hold on
plot(t,stp2,'-.b','LineWidth',2);hold off
set(gcf,'Renderer','Zbuffer');set(gcf,'Color',[1 1 1]);set(gca,'FontSize',14)
xlabel('\itt','FontSize',16)
ylabel('\its\rm(\itt\rm) , \itds\rm(\itt\rm)/\itdt ,
\itd\rm^2\its\rm(\itt\rm)/\itdt\rm^2','FontSize',16)
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It is important to realize that the following three different functions will all have the same derivative as illustrated in Fig. 3.4.

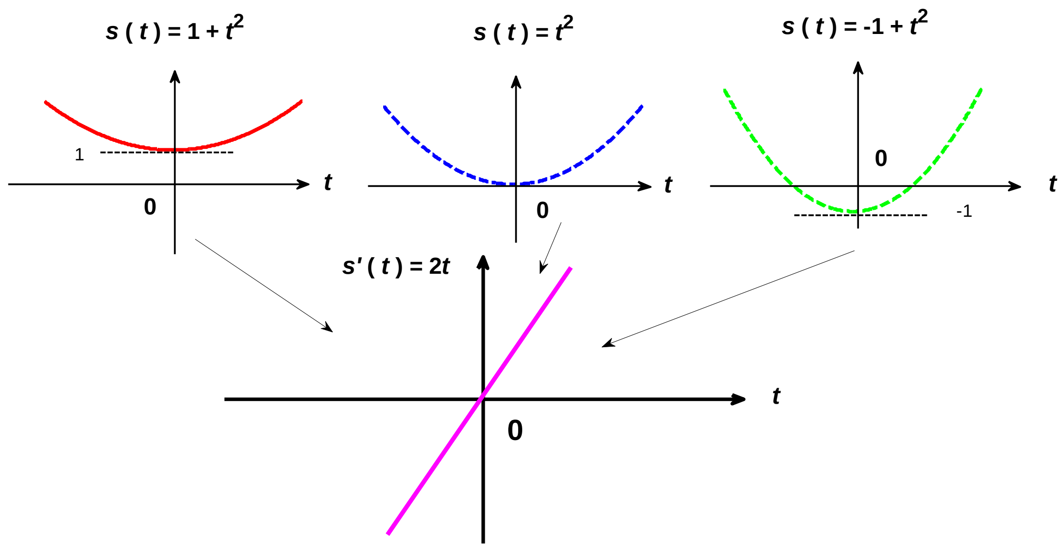

Fig. 3.4 Three different $s(t)$ having the same derivative.

From Fig. 3.4, we gather that given an $s'(t)$, it is impossible to go back to the specific $s(t)$ that led to the given $s'(t)$. This problem can only be solved by using boundary or initial conditions. For instance, as a boundary condition, if it is specified that $s(t=0)=-1$, then it is possible to go back to $s(t)=-1+t^2$.

Now we consider an important form of $s(t)$, that is used often in our discipline, that is

$$s(t)=A_s\cos(2\pi f_s t)=A_s\cos(\omega_s t) \tag{3.6}$$

which is known as a sinusoidal function (waveform), with a frequency $f_s = 1/T_s$ or radial frequency ω_s and peak amplitude A_s. Note that (3.6) could have equally been written using sine instead of cosine.

Differentiating (3.6) with respect to t, we get

$$\frac{d}{dt}s(t) = s'(t) = -2\pi f_s A_s \sin(2\pi f_s t) = -\omega_s A_s \sin(\omega_s t) \tag{3.7}$$

Plotting $s(t)$ and $s'(t)$ given in (3.6) and (3.7) for a time interval $0 \le t \le T_s$, we get the plots shown in Fig. 3.5.

Exercise 3.1 : By setting $s(t)$ to a sine function, i.e., – $s(t)$'yi aşağıda verilen sinus fonksiyonuna dönüştürün

$$s(t) = A_s \sin(2\pi f_s t) = A_s \sin(\omega_s t) \tag{3.8}$$

Find $s'(t)$ and plot $s(t)$ and $s'(t)$ like given in Fig. 3.5. – Şekil 3.5'de gösterildiği gibi, $s(t)$ ve $s'(t)$ 'yi çizin

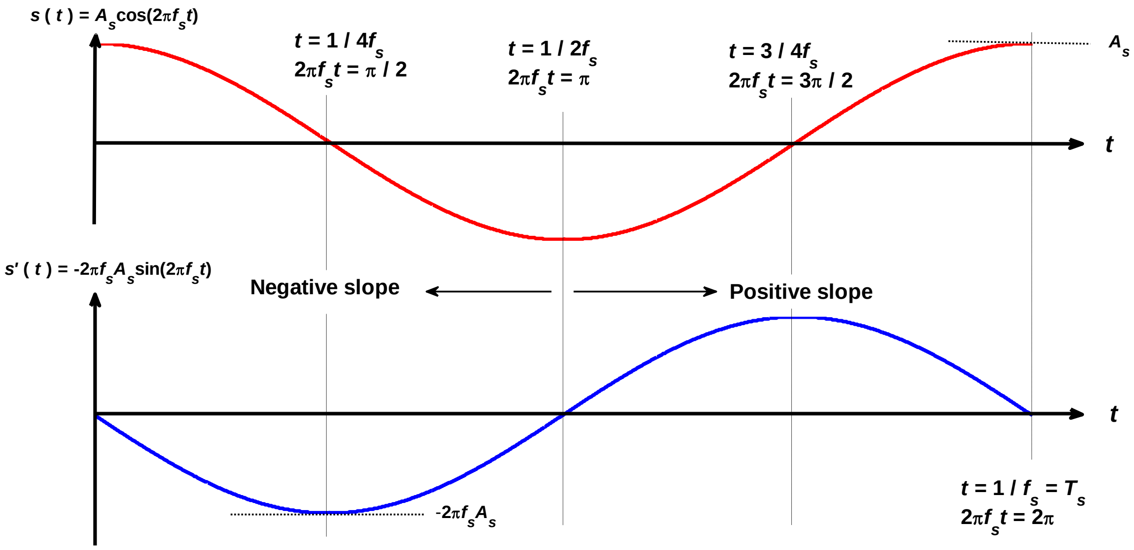

Fig. 3.5 Plots of $s(t)$ and $s'(t)$ given in (3.6) and (3.7) for a time interval $0 \le t \le T_s$.

Note that it is good practice to make time coincident plots as displayed in Fig. 3.5.

Finally in this section, we would like to point out that differentiation also helps to find the degree of dependence of a function on its variable, i.e., how strongly or weakly a function depends on its variable. In this sense consider the following two functions

$$s_1(t) = t^{2.3} \quad , \quad s_2(t) = t^{3.1}$$
$$\frac{d}{dt}s_1(t) = s_1'(t) = 2.3t^{1.3} \quad , \quad \frac{d}{dt}s_2(t) = s_2'(t) = 3.1t^{2.1} \qquad (3.9)$$

After differentiation we see that for positive values of $t > 1$, the slope of $s_2(t)$ will be greater than the slope of $s_1(t)$, which can be mathematically expressed as

$$\text{For } t > 1 \quad \frac{d}{dt}s_2(t) = 3.1t^{2.1} > \frac{d}{dt}s_1(t) = 2.3t^{1.3} \qquad (3.10)$$

The implications of (3.9) and (3.10) are that when compared among themselves, $s_1(t)$ seems to have a weaker dependence on t, whereas $s_2(t)$ has a stronger dependence on t, when $t > 1$. Of course the complete picture can be obtained by plotting and analysing the graphs of $s_1'(t)$ and $s_2'(t)$ against t. This is done in Fig. 3.6.

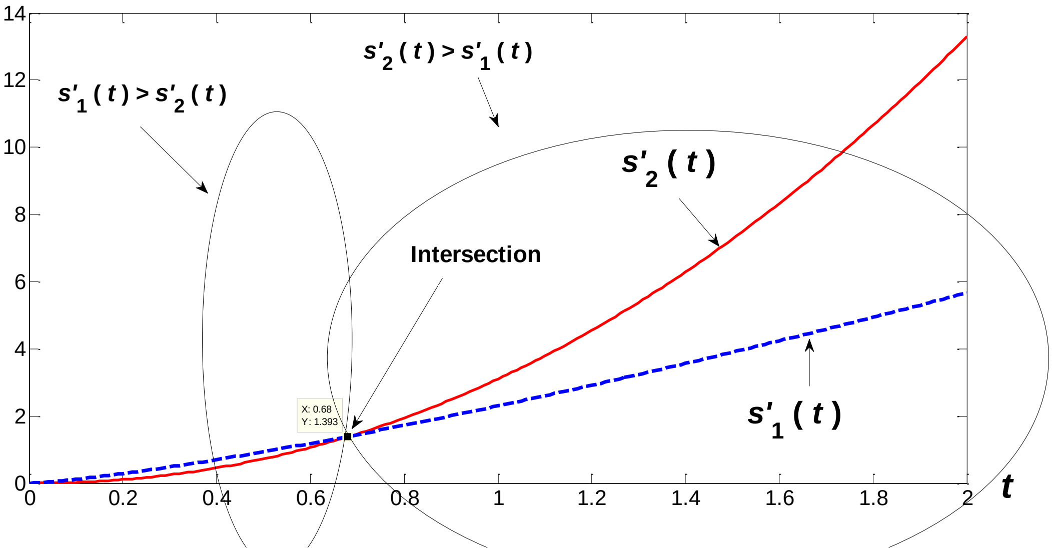

Fig. 3.6 Plots of $s_1'(t)$ and $s_2'(t)$ given in (3.10).

Exercise 3.2 : By considering (3.10) and Fig. 3.6, explain how to find the point of intersection given in Fig. 3.6, that is the intersection between $s_1'(t) > s_2'(t)$ and $s_2'(t) > s_1'(t)$. (3.10) ve Şekil 3.6 göz önüne alarak, Şekil 3.6'daki kesişme noktasının nasıl bulunabileceğini açıklayın.

Now let's turn to a more complicated example and compare the dependence of the following two functions on its argument.

$$s_1(t) = t^n \quad , \quad s_2(t) = \exp(t^n) \quad \text{for} \quad t \geq 0 \quad , \quad n = 2,\ 4,\ 6,\ \cdots$$
$$\frac{d}{dt}s_1(t) = nt^{n-1} \quad , \quad \frac{d}{dt}s_2(t) = nt^{n-1}\exp(t^n) = \frac{d}{dt}s_1(t)\exp(t^n) \qquad (3.11)$$

It is clear from the second line of (3.11) that the dependence of $s_2(t)$ on t is stronger than the dependence of $s_1(t)$ on t for $t > 0$. This means the rise of $s_2(t)$ with $t > 0$ is steeper than that of $s_1(t)$ with t. Again this conclusion can be verified by plotting $s_1(t)$ and $s_2(t)$.

Exercise 3.3 : By taking the following $s_1(t)$ and $s_2(t)$ in the range $t \geq 0$, by examining $s_1^{'}(t)$ and $s_2^{'}(t)$ or directly $s_1(t)$ and $s_2(t)$. Find the degree of dependence of $s_1(t)$ and $s_2(t)$ on t, and make a comparison of these dependencies. If necessary, make plots $s_1(t)$, $s_2(t)$ and $s_1^{'}(t)$, $s_2^{'}(t)$.

$s_1(t)$ ve $s_2(t)$, $s_1^{'}(t)$ ve $s_2^{'}(t)$ $t \geq 0$ aralığında grafiğini çizerek (veya tahminde bulunarak), aşağıdaki verilen $s_1(t)$ ve $s_2(t)$, ve ilaveten $s_1^{'}(t)$ ve $s_2^{'}(t)$ $t \geq 0$ aralığında, bu fonksiyonların, t bağımlılığını inceleyin.

$$s_1(t) = 1 + t^2 \quad , \quad s_2(t) = \frac{1+t^2}{1+t} \tag{3.12}$$

4. Integration of a Function

Basically, integral or integration of a function is finding the area underneath. Mathematically, it is shown as

$$y(t) = \int s(t)\, dt \quad \text{or} \quad y(t) = \int_{t_1}^{t_2} s(t)\, dt \tag{4.1}$$

The first expression in (4.1) is known as indefinite integral, the second (on the right hand side) is called the definite integral. For definite integral, there is no need to insert initial conditions or seek such conditions. In the case of indefinite integral however, initial conditions may be important. This is further explained later.

Now we consider two simple time functions, namely $s_1(t)$ and $s_2(t)$ defined as

$$s_1(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ A & t \geq 0 \end{cases} \qquad s_2(t) = \begin{cases} 0 & \text{for} \quad t < -2 \\ A & t \geq -2 \end{cases} \tag{4.2}$$

Upon performing indefinite integrations of $s_1(t)$ and $s_2(t)$, we find

$$y_1(t) = \int s_1(t)\, dt = At + C_1 \quad , \quad y_2(t) = \int s_2(t)\, dt = At + C_2 \tag{4.3}$$

As seen from (4.3), the time dependent term, i.e., At is common to both $y_1(t)$ and $y_2(t)$. But it is clear from (4.2) that $s_1(t)$ and $s_2(t)$ are different. Hence we expect that difference to be reflected into the integration constants C_1 and C_2. To find C_1 and C_2, we simply apply the boundary or the initial conditions at $t = 0$ for $y_1(t)$ and $t = -2$ for $y_2(t)$, thus

$$y_1(t=0)=0=C_1 \;\;,\;\; \text{thus} \;\; y_1(t)=\begin{cases}0 & \text{for} \;\; t<0\\ At & t\geq 0\end{cases}$$

$$y_2(t=-2)=0=-2A+C_2 \;\;,\;\; C_2=2A \;\;,\;\; \text{thus} \;\; y_2(t)=\begin{cases}0 & \text{for} \;\; t<-2\\ At+2A & t\geq -2\end{cases} \qquad (4.4)$$

As demonstrated by (4.4), there is no accumulated area for $y_1(t)$ prior to $t=0$, whereas for $y_2(t)$, the accumulated area up to $t=0$ is $2A$. Fig. 4.1 contains the graphical details of (4.4).

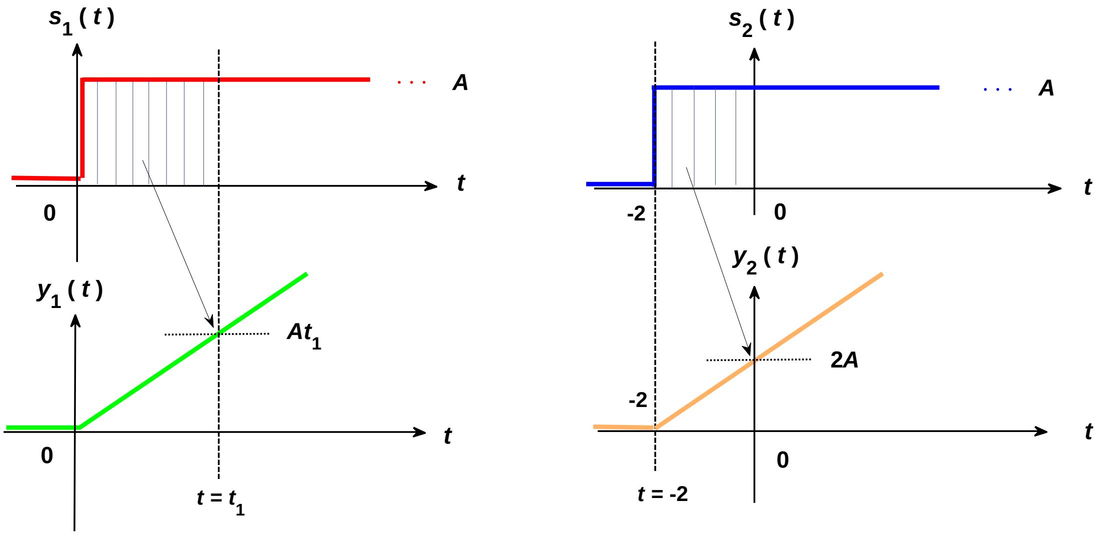

Fig. 4.1 Functions $s_1(t)$ and $s_2(t)$ and their corresponding integrals $y_1(t)$ and $y_2(t)$.

Exercise 4.1 : Repeat the evaluations in (4.3), (4.4) and redraw the plots in Fig. 4.1, by setting $s_1(t)$ and $s_2(t)$ to

$$s_1(t)=\begin{cases}0 & \text{for} \;\; t<-1\\ A & t\geq -1\end{cases} \qquad s_2(t)=\begin{cases}0 & \text{for} \;\; t<2\\ A & t\geq 2\end{cases} \qquad (4.5)$$

5. Sinusoidal Functions

In this section, we examine the general form of the sinusoidal function given by (3.6). The various forms of these functions are written below in (5.1).

$$\begin{aligned} s_c(t)&=A_s\cos(2\pi f_s t) && \leftarrow && \text{Cosine form}\\ s_s(t)&=A_s\sin(2\pi f_s t) && \leftarrow && \text{Sine form}\\ s_p(t)&=A_s\exp(j2\pi f_s t)=A_s e^{j2\pi f_s t} && \leftarrow && \text{Complex form} \end{aligned} \qquad (5.1)$$

The writing in (5.1) assumes that time reference point is taken to be $t=0$. If we want to use a different reference point or indicate the time difference between two signals, then we use time delay or time advance as indicated in (5.2)

$$\begin{array}{c} \text{Time delay} \\ \downarrow \\ s_1(t) = A_s \cos\left[2\pi f_s (t - t_d)\right] \\ s_2(t) = A_s \cos\left[2\pi f_s (t + t_d)\right] \\ \uparrow \\ \text{Time advance} \end{array} \qquad (5.2)$$

An example of the polar coordinate representation of sinusoidal functions is shown in Fig. 5.1. As seen from there, the sinusoidal function of $s_c(t) = A_s \cos(2\pi f_s t) = A_s \cos(\omega_s t) = A_s \cos(\phi_s)$ can be visualized as a phasor (indicated by a blue arrow) circulating around a circle of radius A_s with the radial speed of ω_s, such that at a given time, the angle of this phasor is ϕ_s. This means the time delay of (5.2) can also be expressed by phase delay as stated in (5.3).

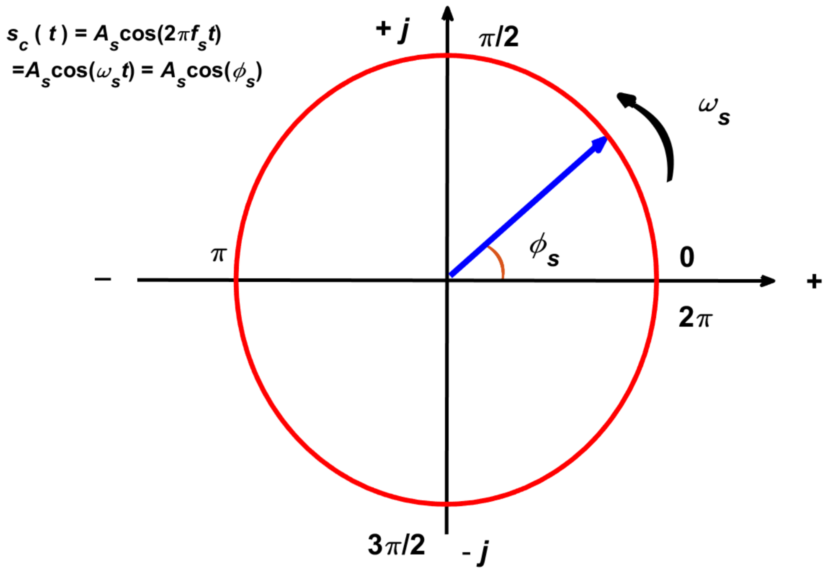

Fig. 5.1 Phasor representation of a sinusoid consisting of a cosine function.

$$s_1(t) = \underbrace{A_s \cos\left[2\pi f_s (t - t_d)\right]}_{\text{Time delay}} = A_s \cos(2\pi f_s t - 2\pi f t_d) = \underbrace{A_s \cos(2\pi f_s t - \phi_d)}_{\text{Phase delay}} \qquad (5.3)$$

In order to fully describe the meaning of the phase difference term, ϕ_d, we take two cosine time functions as follows

$$\begin{array}{ll} s_1(t) = A_s \cos(2\pi f_s t) = A_s \cos(\phi_s) = A_s \cos\left[\phi_s(t)\right] & \leftarrow \text{Cosine function with no phase delay} \\ s_2(t) = A_s \cos(2\pi f_s t - \phi_d) = A_s \cos(\phi_s - \phi_d) & \leftarrow \text{Cosine function with phase delay of } \phi_d \end{array} \qquad (5.4)$$

$s_1(t)$ and $s_2(t)$ given in (5.4) can be plotted in a phasor diagram as shown in Fig. 5.2.

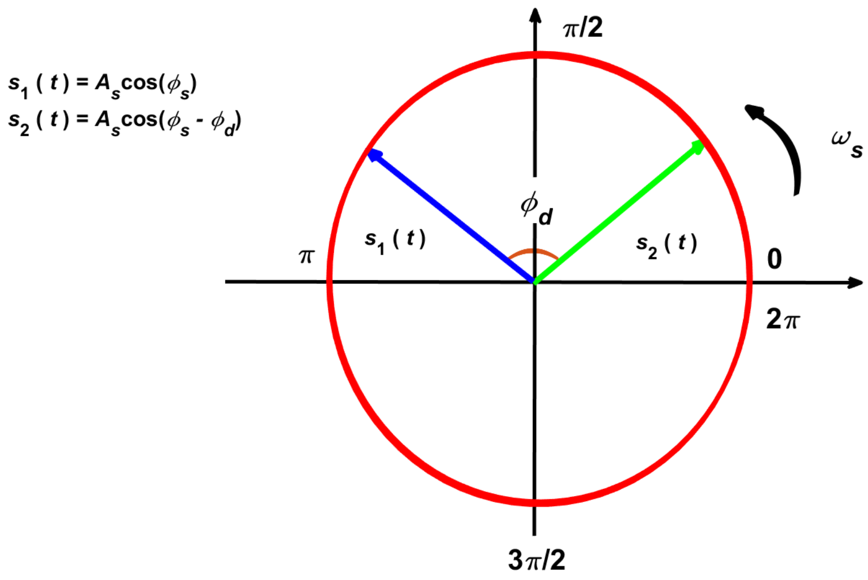

Fig. 5.2 Phasor diagram illustrating the phase difference between $s_1(t)$ and $s_2(t)$.

The situation from the point of view of cosine time functions of (5.4) is displayed in Fig. 5.3.

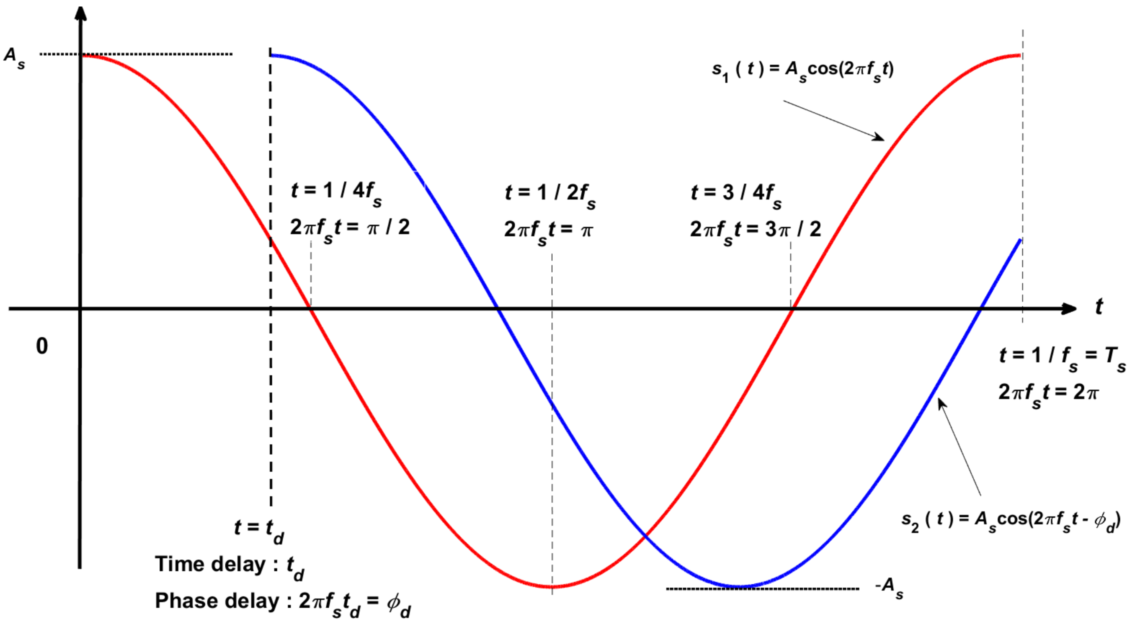

Fig. 5.3 Two cosine time functions of (5.4) highlighting the physical appearance of time and phase delays.

Now we wish to prove that for sinusoidal functions, all time delays can be reduced to a phase delay confined to the angular interval of $0 \rightarrow 2\pi$. For this we set $s_1(t)$ and $s_2(t)$ as follows

$$s_1(t) = A_s \cos(2\pi f_s t)$$
$$s_2(t) = A_s \cos[2\pi f_s (t - t_d)] = A_s \cos(2\pi f_s t - \phi_d)$$
$$\text{Time delay : } t_d = \frac{1}{f_s} + \frac{1}{f_s} + \frac{1}{2 f_s} = \frac{2.5}{f_s} = 2.5 T_s$$
$$\text{Phase delay : } 2\pi f_s t_d = \phi_d = 5\pi \tag{5.5}$$

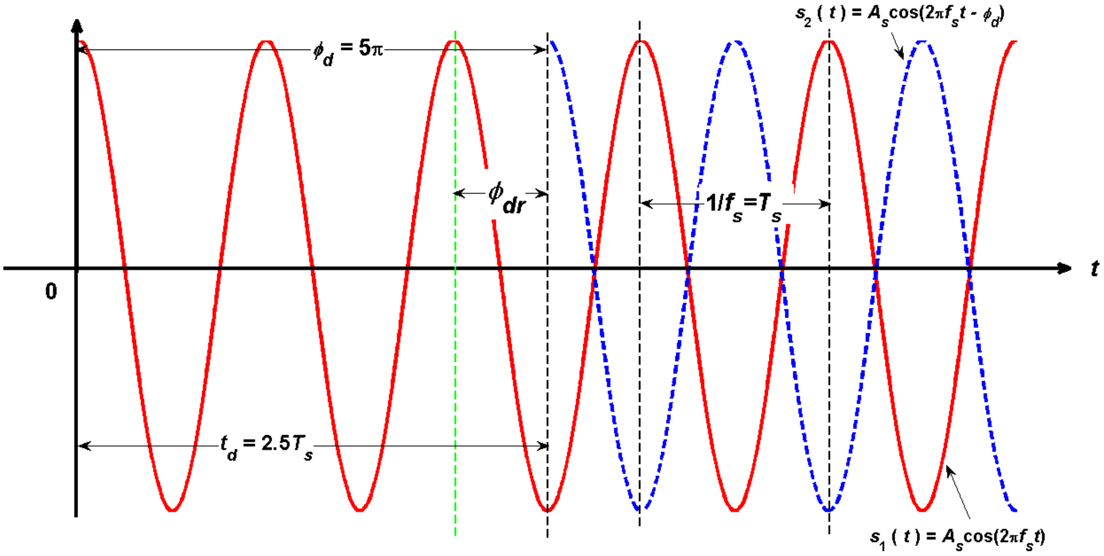

Fig. 5.4 Illustration of time and phase delays for (5.5).

In Fig. 5.4, $s_1(t)$ and $s_2(t)$ are plotted such that the actual phase difference (measured from $t = 0$) between $s_1(t)$ and $s_2(t)$ is $\phi_d = 5\pi T_s$, but if we measure between the nearest points of $s_1(t)$ and $s_2(t)$, where the same action is repeated (in Fig. 5.4, this is taken to be the adjacent positive peaks of $s_1(t)$ and $s_2(t)$) then this phase difference reduces to $\phi_d = \pi$. Therefore our problem boils down to proving the following equality

$$A_s \cos[2\pi f_s (t - t_d)] = A_s \cos(2\pi f_s t - \phi_{dr}) \tag{5.6}$$

Now by using the identity

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B) \tag{5.7}$$

(5.6) will become

$$\cos(2\pi f_s t)\overbrace{\cos\left(2\pi f_s \times \frac{2.5}{f_s}\right)}^{-1} + \sin(2\pi f_s t)\overbrace{\sin\left(2\pi f_s \times \frac{2.5}{f_s}\right)}^{0} = \cos(2\pi f_s t)\overbrace{\cos(\pi)}^{-1} + \sin(2\pi f_s t)\overbrace{\sin(\pi)}^{0}$$
$$\cos(2\pi f_s t) \equiv \cos(2\pi f_s t) \tag{5.8}$$

Hence the proof that time delay is equivalent to the smallest phase delay in the interval of $0 \rightarrow 2\pi$ is completed.

These notes are based on

1) MATLAB m files.
2) My own Lecture Notes.